

Related topics

Spiral spring, gravity pendulum, spring constant, torsional vibration, torque, beat, angular velocity, angular acceleration, characteristic frequency.

Principle

Two equal gravity pendula with a particular characteristic frequency are coupled by a “soft” spiral spring. The amplitudes of both pendula are recorded as a function of time for various vibrational modes and different coupling factors using Cobra4. The coupling factors are determined by different methods.

Equipment

2 Pendulum w. recorder connection	02816-00	2 Support rod PHYWE, square, $l = 1000$ mm	02028-55
1 Helical spring, 3 N/m	02220-00	2 Right angle clamp PHYWE	02040-55
1 Rod with hook	02051-00	1 Measuring tape, $l = 2$ m	09936-00
1 Weight holder f. slotted weights	02204-00	4 Connecting cord, $l = 1000$ mm, red	07363-01
5 Slotted weight, 10 g, black	02205-01	4 Connecting cord, $l = 1000$ mm, blue	07363-04
2 Capacitor, 10 μ F/35 V	39105-28	1 Software Cobra4 - multi-user licence	14550-61
1 Cobra4 Wireless Manager	12600-00		
2 Cobra4 Wireless-Link	12601-00		
2 Cobra4 Sensor-Unit Electricity	12644-00		
1 Power supply 0-12 V DC/6 V, 12 V AC	13505-93	Additionally required	
2 Bench clamp PHYWE	02010-00	PC with USB-port, Windows XP or higher	

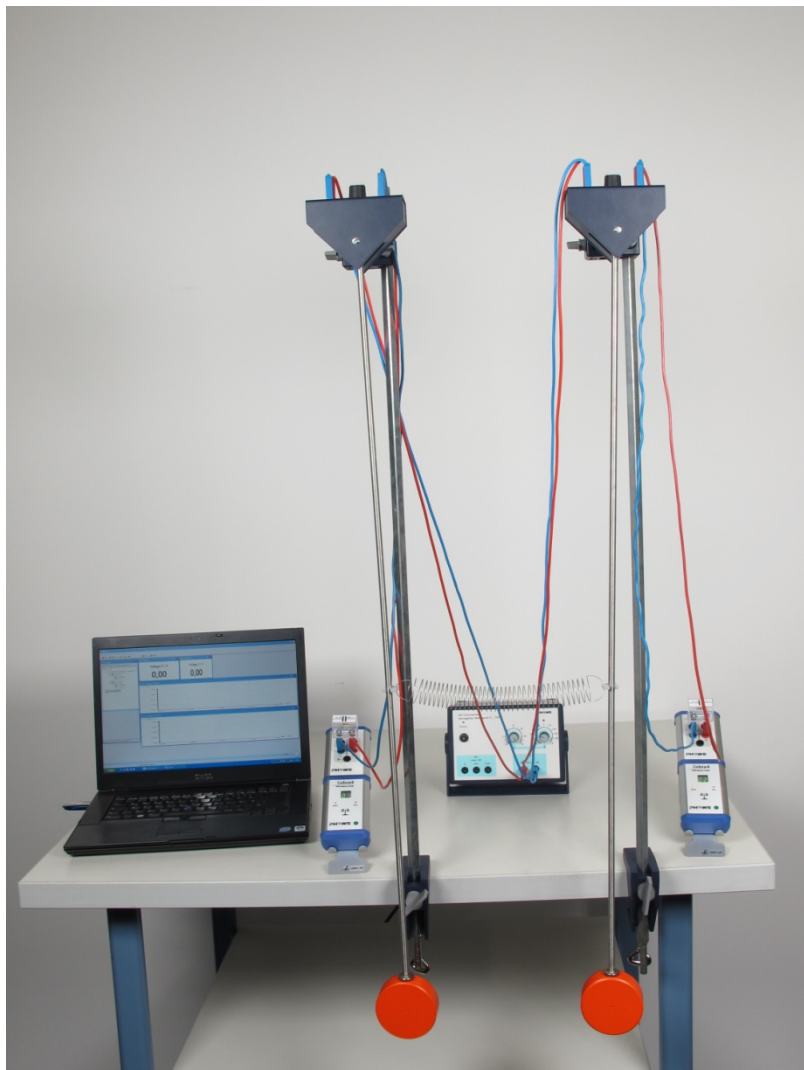


Fig. 1: Experimental set-up.

Tasks

- To determine the spring constant of the coupling spring.
- To determine and to adjust the characteristic frequencies of the uncoupled pendula.
- To determine the coupling factors for various coupling-lengths using
 - the apparatus constants.
 - the angular frequencies for “- inphase” and “in opposite phase” vibration.
 - the angular frequencies of the beat mode.
- To check the linear relation between the square of the coupling lengths and
 - the particular frequencies of the beat mode.
 - the square of the frequency for “in opposite phase” vibration.
- To determine the pendulum’s characteristic frequency from the vibrational modes with coupling and to compare this with the characteristic frequency of the uncoupled pendula.

Set-up and procedure

Before measurement can begin, the exact value of the spring constant D_F of the coupling spring has to be determined. A supporting rod is fixed to the edge of table by means of a bench clamp. The spring is suspended on the rod from a hook which is attached to the supporting rod via a right angle clamp. Applying Hook’s law

$$F = -D_F x$$


the spring constant D_F can be calculated if the extension x of the spring is measured for different slotted weights attached to the spring. The pendula are then set up without coupling springs as shown in Fig. 1. The input sockets of the pendula are now switched in parallel to the DC-output of the power supply unit. The yellow output sockets of the pendula are connected to the Cobra4 Sensor-Units Electricity. The DC-output voltage of the power supply unit is adjusted to 10 V. To set the pendula into vibration the pendula rods are touched with the finger-tips on their upper third and simultaneously moved to and fro till the desired amplitudes have been established. In this way transverse vibrations can be avoided. In view of the subsequent experiments with coupled pendula care should be taken already at this stage to ensure that the pendula are oscillating in the same plane. From the plotted curves the period T_0 is determined several times for each pendulum. The mean values of the periods, $\overline{T_0}$, of both pendula have to be identical within the limits of error. If deviations are observed, the lengths of the pendulum rods have to be adjusted. This is done by detaching the counter nut on the threaded rod of the pendulum weight, adjusting the pendulum length and manually retightening the counter nut. For the performance of the experiments with coupled pendula, the coupling spring is fixed to the plastic sleeves on the pendulum rods at a point equidistant from the pendulum’s fulcrum. Furthermore the “zero”-positions have to be readjusted. It has to be insured that there is no electric conductivity between the pendula.

The amplitudes as a function of time are to be recorded for different coupling lengths l using the following initial conditions:

- Both pendula are deflected with the same amplitude to the same side and simultaneously released. (“in- phase” vibration).
- Both pendula are deflected with the same amplitude but in opposite directions and simultaneously released. (“in opposite phase” vibration).
- One pendulum remains at rest. The second pendulum is deflected and released (beat mode). Here satisfactory results can only be achieved if during the preparation both pendula have been properly adjusted in such a way that they have in fact the same period $\overline{T_0}$.

In all three cases the vibrations have to be recorded for at least three or four minutes. From the plotted curves the mean values for the corresponding vibrational periods can be determined

Connect the Cobra4 Wireless Manager to the USB-interface of the computer and plug the Cobra4 Sensor-Units Electricity on the Cobra4 Wireless-Links. Start the “measure”-software and open the “Coupled

Pendula” experiment. (“Experiment” → “Open experiment”). All pre-settings that are necessary for measured value recording are now carried out. Click on  in the icon strip to start measurement.

Theory and evaluation

If two gravity pendula P1 and P2 with the same angular characteristic frequency ω_0 are coupled by a spring, for the position of rest and small angle deviation \sim due to the presence of gravity and spring-tension we have the following torques (Fig. 2):

torque due to gravity:

$$M_{s,0} = m g L \sin \phi_0 \sim m g L \phi_0 \quad (1)$$

torque due to spring-tension:

$$M_{FD} = - D_F x_0 l \cos \phi_0 \sim - D_F x_0 l$$

D_F = spring constant

x_0 = extension of the spring

l = coupling length

m = pendulum mass

L = pendulum length

g = acceleration due to gravity

ϕ_0 = angle between the vertical and the position of rest

If P₁ is now deflected by ϕ_1 and P₂ by ϕ_2 (see Fig. 3) and subsequently released, we have because of

$$I \ddot{\phi} = M$$

I = moment of inertia of a pendulum around its fulcrum

$$I \ddot{\phi}_1 = M_1 = - m g L \phi_1 + D_F l^2 (\phi_2 - \phi_1) \quad (2)$$

$$I \ddot{\phi}_2 = M_2 = - m g L \phi_2 + D_F l^2 (\phi_2 - \phi_1)$$

Introducing the abbreviations

$$\omega_0^2 = \frac{m g L}{I} \quad \text{and} \quad \Omega^2 = \frac{D_F l^2}{I} \quad (3)$$

we obtain from Eqs. (2)

$$\ddot{\phi}_1 + \omega_0^2 \phi_1 = - \Omega^2 (\phi_2 - \phi_1) \quad (4)$$

$$\ddot{\phi}_2 + \omega_0^2 \phi_2 = + \Omega^2 (\phi_2 - \phi_1)$$

At $t = 0$ the following three initial conditions are to be realized successively.

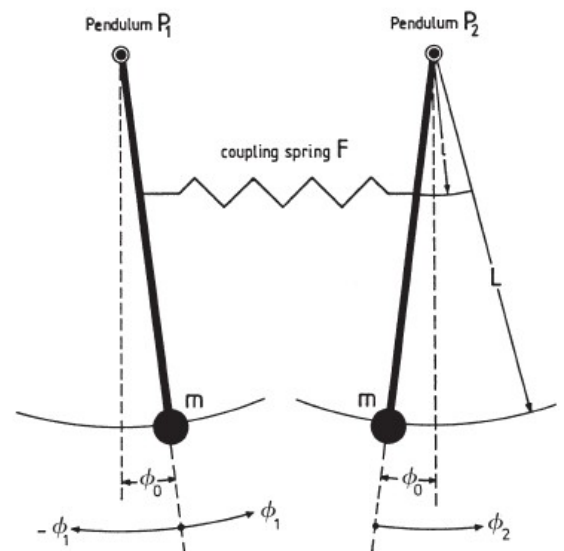


Fig. 2: Diagram of coupled pendula in rest.

A: "inphase" vibration

$$\phi_1 = \phi_2 = \phi_A; \phi_1 - \phi_2 = 0$$

B: "in opposite phase" vibration

$$-\phi_1 = \phi_2 = \phi_A; \phi_1 - \phi_2 = 2\phi_A \quad (5)$$

C: beat mode

$$\phi_1 = \phi_A; \phi_2 = 0; \phi_1 - \phi_2 = \phi_A$$

The general solutions of the system of differential equations (4) with the initial conditions (5) are:

$$\text{A: } \phi_1(t) = \phi_2(t) = \phi_A \cos \omega_0 t \quad (6a)$$

$$\text{B: } \phi_1(t) = \phi_A \cos \left(\sqrt{\omega_0^2 + 2 \Omega^2} t \right) \quad (6b)$$

$$\phi_2(t) = -\phi_A \cos \left(\sqrt{\omega_0^2 + 2 \Omega^2} t \right)$$

$$\text{C: } \phi_1(t) = \phi_A \cos \left(\frac{\sqrt{\omega_0^2 + 2 \Omega^2} - \omega_0}{2} \cdot t \right) \quad (6c)$$

$$\cdot \cos \left(\frac{\sqrt{\omega_0^2 + 2 \Omega^2} + \omega_0}{2} \cdot t \right)$$

$$\phi_2(t) = -\phi_A \sin \left(\frac{\sqrt{\omega_0^2 + 2 \Omega^2} - \omega_0}{2} \cdot t \right)$$

$$\cdot \sin \left(\frac{\sqrt{\omega_0^2 + 2 \Omega^2} + \omega_0}{2} \cdot t \right)$$

Comment

A: "inphase" vibration Both pendula vibrate inphase with the same amplitude and with the same frequency ω_0 . The latter is identical with the angular characteristic frequency ω_0 of the uncoupled pendula.

$$\omega_0 = \omega_0 \quad (7a)$$

B: "in opposite phase" vibration Both pendula vibrate with the same amplitude and with the same frequency ω_c but there is a phase-difference of π . In accordance with (3), the angular frequency

$$\omega_c = \sqrt{\omega_0^2 + 2 \Omega^2} \quad (7b)$$

depends on the coupling length l .

C: Beat mode For weak coupling, e.g. $\omega_0 \gg \Omega$, the angular frequency of the first factor can be expressed as follows:

$$\omega_1 = \frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \approx \frac{\Omega^2}{2\omega_0} \quad (8a)$$

For the angular frequency of the second factor we get:

$$\omega_2 = \frac{\sqrt{\omega_0^2 + 2\Omega^2} + \omega_0}{2} \approx \omega_0 + \frac{\Omega^2}{2\omega_0} \quad (8b)$$

Subsequently we get:

$$\omega_1 < \omega_2$$

Fig. 3 shows the amplitudes $\Phi_1(t)$ and $\Phi_2(t)$ of both pendula as a function of time for the beat case and for different coupling lengths l .

As coupling factor we define the ratio

$$K = \frac{D_f l^2}{mgL + D_f l^2} \quad (9)$$

From Eq. (3) and Eq. (9) we get

$$K = \frac{\Omega^2}{\omega_0^2 + \Omega^2} \quad (10)$$

The coupling factor K of Eq. (10) can be calculated from the frequencies of the individual vibrational modes.

Substituting Eq. (7a) and Eq. (7b) into Eq. (10) results in

$$K = \frac{\omega_c^2 - \omega_g^2}{\omega_c^2 + \omega_g^2} \quad (11)$$

(“in opposite phase” vibration)

Substituting Eq. (8a) and Eq. (8b) into Eq. (10) yields:

$$K = \frac{2\omega_1\omega_2}{\omega_1^2 + \omega_2^2} \quad (12)$$

(beat case)

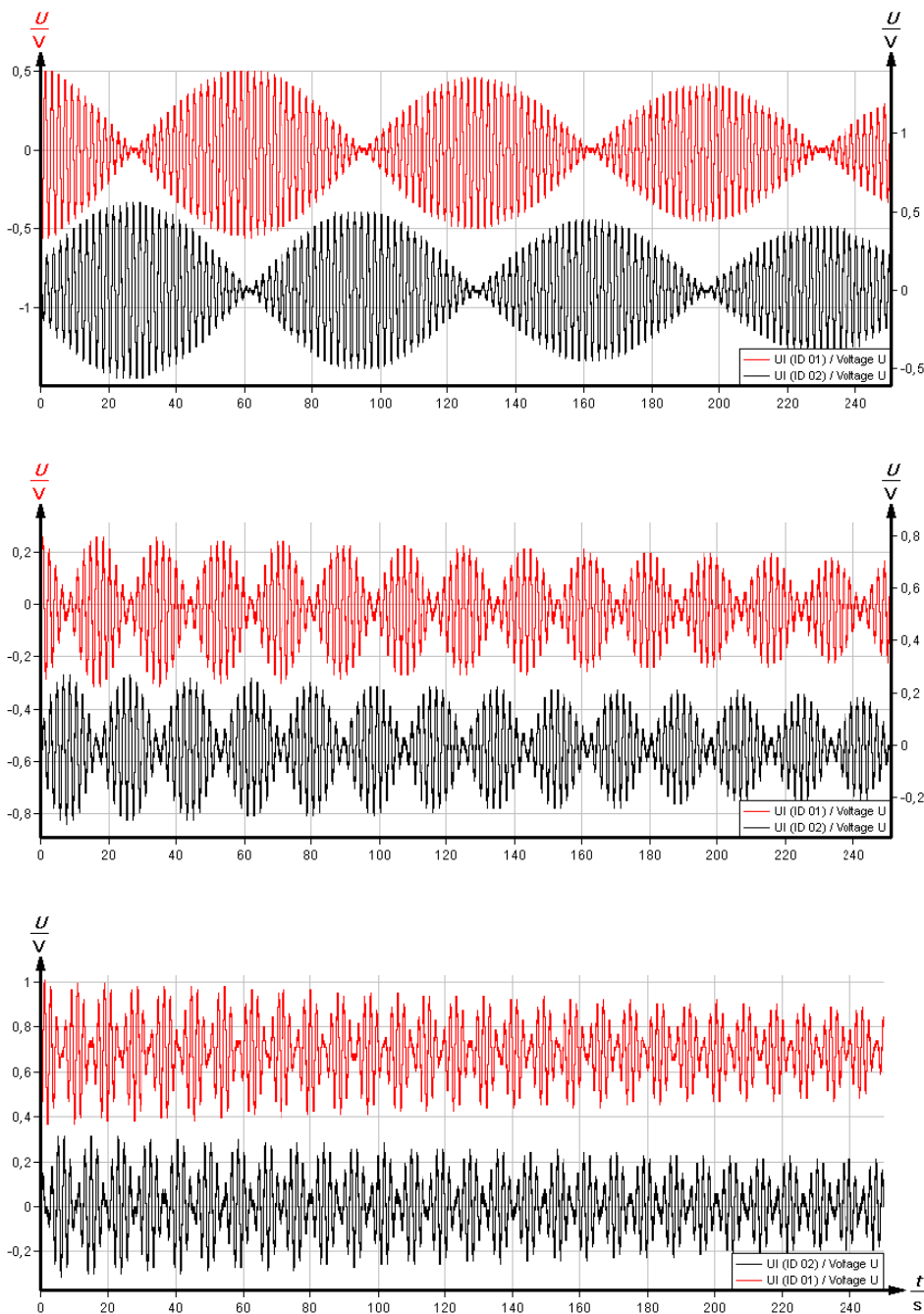


Fig. 3: Amplitude curves of the vibrations of coupled pendula in the beat mode for three different coupling lengths l (30 cm, 60 cm and 90 cm) as a function of time.

To check the influence of coupling length on the frequencies of the individual vibrational modes, we substitute Eq. (11) and Eq. (12) into Eq. (9). Then we get for the in opposite phase vibration:

$$\omega_1^2 = \frac{2D_F\omega_0^2}{mgL} l^2 + \omega_0^2 \quad (13)$$

And for the beat mode:

$$\omega_1 = \omega_0 \frac{D_F}{2mgL} l^2 \quad (14)$$

as well as

$$\omega_2 = \omega_0 \frac{D_F}{2mgL} l^2 + \omega_0 \quad (15)$$

The measurement of the “inphase” vibration of the uncoupled pendula results in the following:

$$\bar{T}_0 = (1.935 \pm 0.003)s; \quad \frac{\Delta\bar{T}_0}{\bar{T}_0} = \pm 0.15\%$$

or

$$\frac{2\pi}{\bar{T}_0} = \bar{\omega}_0 = (3.247 \pm 0.005)s^{-1}$$

Analysis of the measurement

To determine the frequency of the in opposite phase vibration select in the “Analysis” window the option “Fourier analysis” for the measured channel (see Fig. 4):

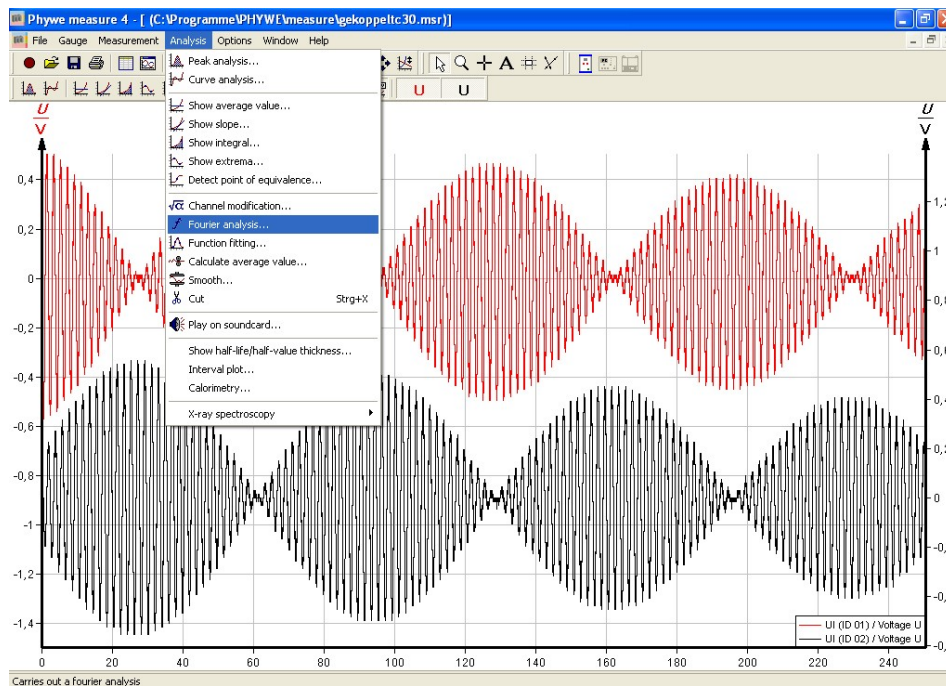


Fig. 4: Fourier analysis of the beat mode.

With the function “Survey” determine then the frequency f_c (see Fig. 5). In our example $f_c = 0.519$ Hz for the coupling length $l = 80$ cm and $\omega_c = 2\pi f_c = 3.261$ 1/s.

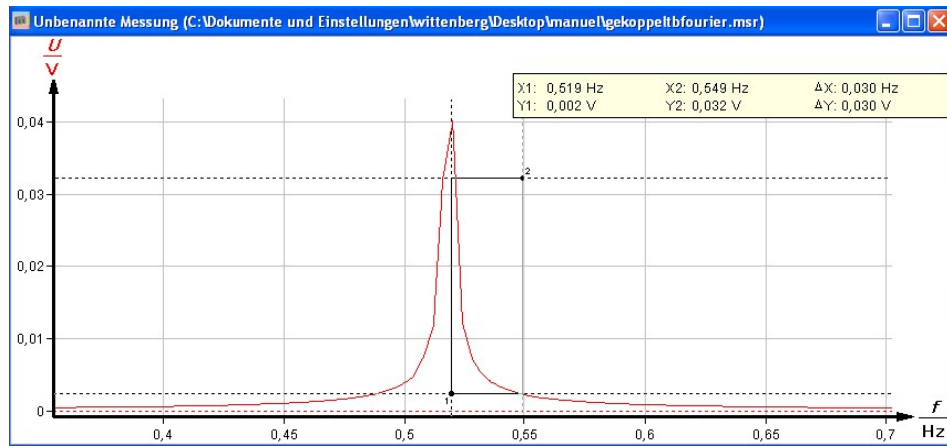
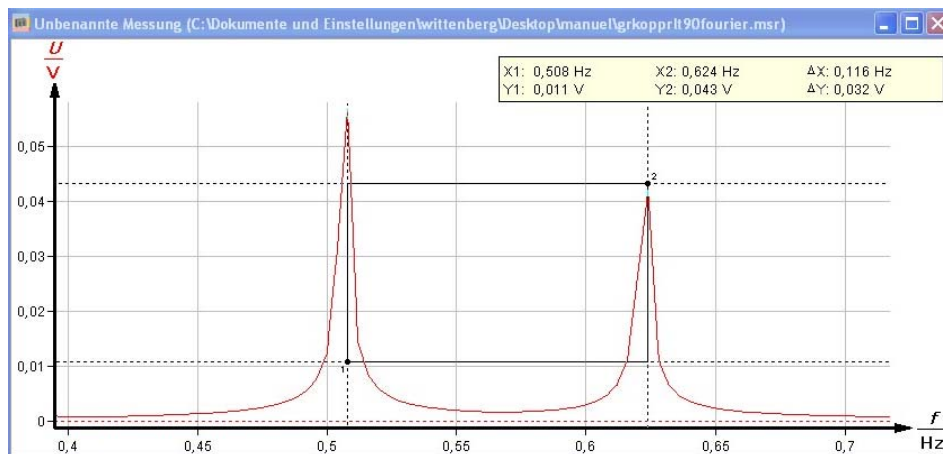


Fig. 5: Determination of the frequency of the opposite phase vibration.

With the same procedure (“Analysis” / “Fourier analysis” / “Survey” function) determine the frequencies ω_1 and ω_2 for the beat case. In our example (see Fig. 6) we obtain for the coupling length $l = 90$ cm: $f_1 = 0.116$ Hz, $\omega_1 = \pi f_1 = 0.364$ 1/s and $\omega_2 = \nu_1 + 2 \pi f_0 = 3.556$ 1/s, with $f_0 = 0.508$ Hz.

Fig. 6: Determination of the frequencies ω_1 and ω_2 for the beat mode.

We used:

$D_F = 3.11$ N/m (measured value)

$L = L_1 = L_2 = 101.5$ cm

(distance fulcrum center of pendulum weight)

$m = 1$ kg

(mass of pendulum rod is not included)

$g = 9.81$ m/s

In Fig. 7 the measured values ω_c^2 of different lengths have been plotted versus l^2 . From the regressive line

$y = A + Bx$

we obtain

$$A = (9.65 \pm 0.58) \text{ s}^{-2}; \frac{\Delta A}{A} = \pm 6\%$$

$$B = (4.81 \pm 0.14) \text{ s}^{-2} \text{ m}^{-2}; \frac{\Delta B}{B} = \pm 3\%$$

Comparison with Eq. (13) gives

$$\sqrt{A} = \omega_0 = (3.106 \pm 0.093) \text{ s}^{-1}; \quad \frac{\Delta\omega_0}{\omega_0} = \pm 3\%$$

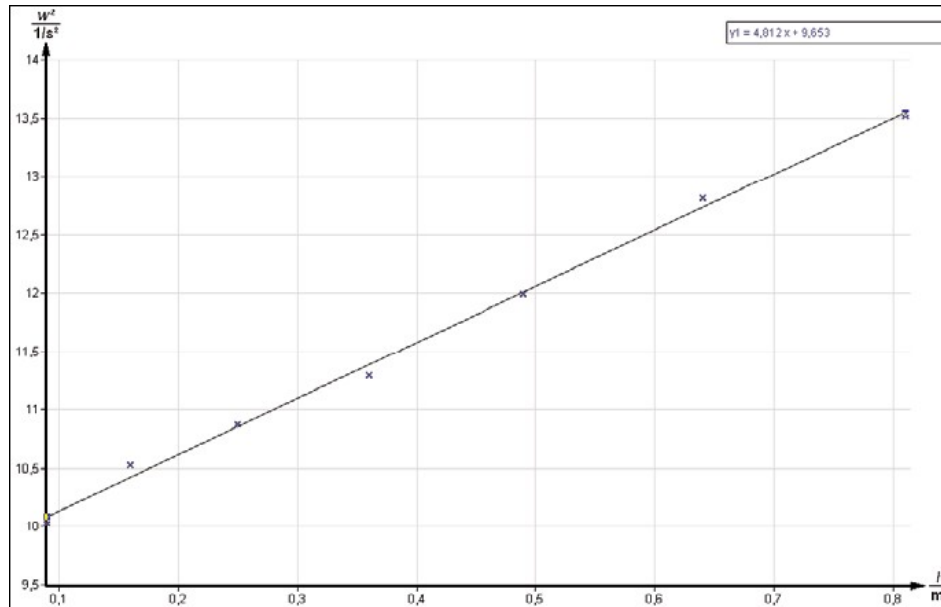


Fig. 7: Frequency of the opposite phase vibration ω_c^2 as a function of the coupling length l^2 .

In Fig. 8 the measured values ω^2 of Tab. 1 have been plotted versus l^2 . The regression line $y = A + Bx$ should confirm Eq. (15). We obtain:

$$A = \omega_0 = (3.099 \pm 0.009) \text{ s}^{-1}; \quad \frac{\Delta\omega_0}{\omega_0} = \pm 3 \%$$

$$B = (0.374 \pm 0.015) \text{ s}^{-1} \text{ m}^{-2}; \quad \frac{\Delta B}{B} = \pm 4 \%$$

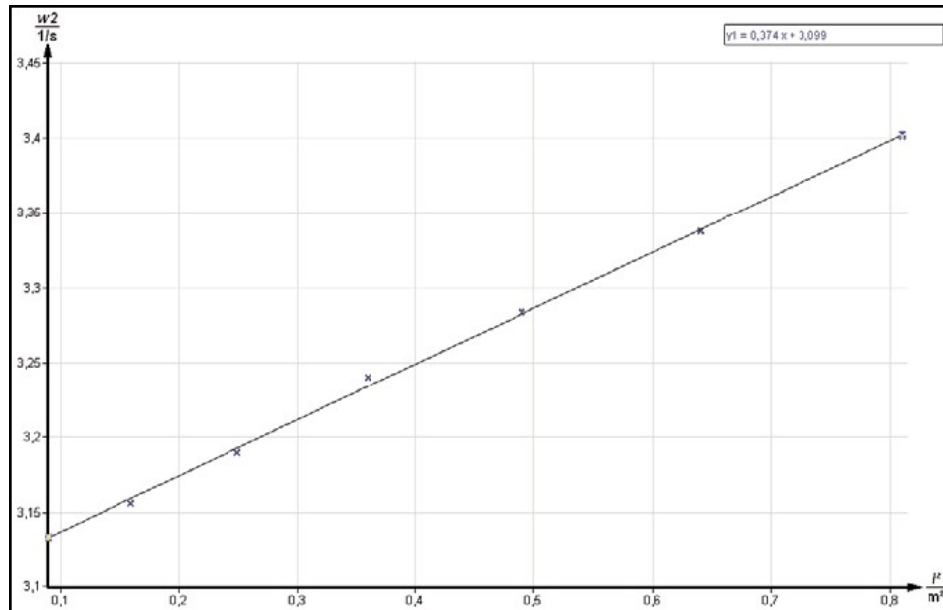


Fig. 8: Frequency ω_2 (beat mode) as a function of the coupling length l^2 .

In Fig. 9 the measured values ω_1 of Tab. 1 are plotted as a function of l^2 . The regression line $y = Bx$ through the origin confirms Eq. (14), $B = (0.374 \pm 0.015) \text{ 1/s}^2 \text{ m}^2$. The results obtained for ω_0 using three different vibrational modes for the coupled pendula are in good agreement with the angular characteristic frequency of the uncoupled pendula.

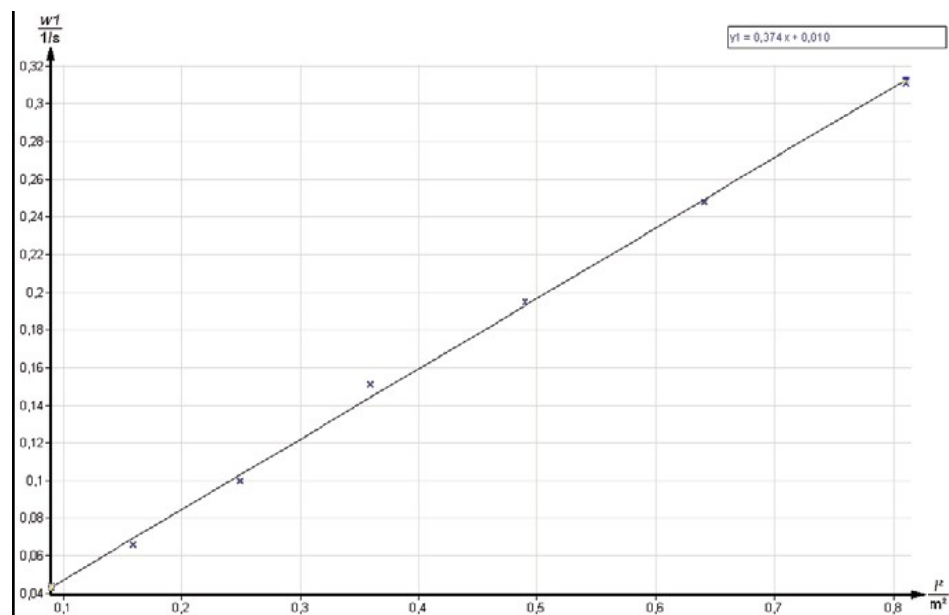


Fig. 9: Frequency ω_1 (beat mode) as a function of the coupling length l^2 .